EFFECT OF THE NONLINEAR ELECTROSTATIC ACTUATION FORCE ON THERMOELASTIC DAMPING/QUALITY FACTOR IN MEMS

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ABSTRACT

MEMS (microelectromechanical systems) with very high quality factors are essential for several applications like ultra-fast and high precision actuators and sensors. Thermoelastic damping is a fundamental dissipation mechanism that always imposes an upper limit on the quality factor of these devices and needs to be understood properly for the accurate prediction of the quality factor. The general theory of thermoelastic damping developed by Zener [**1**] and later modified by Lifshitz and Roukes [**2**] has been used extensively for studying thermoelastic damping in MEMS. The general theory is applicable for MEMS beams undergoing simple harmonic oscillations in the flexural mode. However, under electrostatic actuation, which is the most popular mode of actuation in MEMS, the nature of the thermoelastic damping in MEMS can be significantly different from that predicted by the general theory of thermoelastic damping. This is due to the nonlinear coupling between the electrostatic force and the displacement of the microstructure giving rise to complex oscillations. The general theory of thermoelastic damping is modified in this paper for predicting thermoelastic damping in MEMS under arbitrary electrostatic actuation forces.

INTRODUCTION

MEMS devices are being developed for a variety of applications like accelerometers [**3**], inertial sensors [**4**], chemical sensors [**5**] and RF filters/oscillators [**6**]. In most of these applications, having a high quality factor results in reduced readout errors, lower power requirements, improved stability and increased sensitivity. Thermoelastic damping (TED) [**1**] is one of the fundamental dissipation mechanisms that is inherent to the system (cannot be completely eliminated by improved design or fabrication) and is found to impose an upper limit on the quality factor of MEMS devices [**2**]. The effect of thermoelastic damping on the quality factor of MEMS devices has been studied extensively using both the general theory of thermoelastic damping [**1, 2**] for simple MEMS structures like microbeams [**7**] and also using finite element based physical level (thermomechanical) models for MEMS devices with complex geometries like gyroscopes [**8**], where the general theory (applicable for beams) is not accurate. In this paper, thermoelastic damping in MEMS under electrostatic actuation (the most commonly used mode of actuation in MEMS) is studied. The nonlinear electrostatic force can give rise to complex (non-simple harmonic) oscillations under normal operating conditions [**9**] for which cases thermoelastic damping cannot be predicted correctly using the general theory of TED. The general theory of TED is modified in this paper for the accurate prediction of thermoelastic damping/quality factor in MEMS under arbitrary electrostatic actuation. This modified theory can be conveniently (fast and accurate) used in the design and analysis of MEMS devices.

GENERAL THEORY OF THERMOELASTIC DAMPING

The general theory of thermoelastic damping developed by Zener [**1**] and later modified by Lifshitz and Roukes [**2**], has been used extensively to study the effect of TED on the quality factor of MEMS beams and is found to give good agreement with experimental results for simple harmonic oscillations in the flexural mode of the MEMS beams [**7**], [**10**]. According to the Zener's theory the quality factor due to thermoelastic damping of a beam undergoing simple harmonic motion in the flexural mode, Q_{TED} , is given by [**1**]

$$Q_{TED} = \frac{\hat{C}_p}{E\alpha^2 T_0}\frac{1+(\omega\tau_z)^2}{\omega\tau_z} \quad (1)$$

where E is the Young's modulus, α is the coefficient of thermal expansion, T_0 is the absolute temperature; $\hat{C}_p$ is the specific heat under constant pressure of the beam material and ω is the angular frequency of excitation. τ_z is the relaxation time of the first mode of vibration given by $\tau_z = b^2/(\pi^2\kappa)$ [**1**], where κ is the thermal diffusivity of the beam material and b is the beam thickness. Thus the expression $1/Q_{TED}$ (measure of the amount of energy dissipated) has a maximum value of $E\alpha^2T_0/(2\hat{C}_p)$ when $\omega\tau_z = 1$. Zener's theory (Eq. 1) was made more accurate in [**2**] by Lifshitz and Roukes using the 1-D beam theory and 1-D heat conduction equation. The equation of motion for a beam under thermoelastic damping is given by [**2**]

$$\rho A\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 U}{\partial x^2} + E\alpha I_T\right) = 0 \quad (3)$$

where ρ is the density of the beam material, A and I are the cross-sectional area and the mechanical contribution to the moment of inertia of the beam, respectively and U is the displacement of the beam in the y-direction. x axis is defined along the length of the beam and y and z axes are along the thickness and width direction of the beam, respectively. The term I_T (the thermal contribution to the moment of inertia) is given by

$$I_T = \int_A y\theta dydz \quad (4)$$

where $\theta = T - T_0$ is the change in the temperature from the equilibrium temperature T_0. The linearized heat equation (assuming $\theta \ll T_0$) along the y-direction (temperature gradients along the other directions are assumed to be negligible) is given by [**2**]

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial y^2} + y\frac{E\alpha T_0}{\hat{C}_p}\frac{\partial}{\partial t}\left(\frac{\partial^2 U}{\partial x^2}\right) \quad (5)$$

The coupled thermoelastic equations (Eqs. 3-5) are solved by assuming simple harmonic motion and substituting

$$U(x,t) = U_1(x)e^{i\omega t} \quad \text{and} \quad \theta(x,y,t) = \theta_1(x,y)e^{i\omega t} \quad (6)$$

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into Eq. 5 and getting the temperature profile along the beam $\theta_I(x,y)$. The expression for the temperature profile is next substituted into Eq. 3 and a frequency dependent Young's modulus E_ω is obtained. The quality factor can be computed from the real and the imaginary parts of the E_ω (see [**2**] for details) as

$$Q_{TED} = \frac{\hat{C}_p}{E\alpha^2 T_0}\left(\frac{6}{\xi^2} - \frac{6}{\xi^3}\frac{\sinh\xi + \sin\xi}{\cosh\xi + \cos\xi}\right)^{-1} \tag{7}$$

where $\xi = b[\omega/(2\kappa)]^{1/2}$. Both these theories (the Zener's theory and the Lifshitz and Roukes's theory) are based on the assumption that the motion of the beam is simple harmonic and would fail to predict accurately the thermoelastic damping in beams under complex oscillations (non simple harmonic motions) which can be present under electrostatic actuation.

THERMOELASTIC DAMPING UNDER ELECTROSTATIC ACTUATION: MODIFIED THEORY

Electrostatic actuation in MEMS is realized by applying a potential difference (combination of a dc and an ac voltage) between the microstructure (for example, a micro-beam) and the ground plane. The electrostatic force generated is nonlinear in nature and can give rise to complex nonlinear oscillations of the microstructure [**9**], [**11**]. A modified theory is presented here to study the effect of these complex oscillations on thermoelastic damping/quality factor in MEMS under electrostatic actuation. The modified theory solves the coupled thermoelastic equations (Eqs. 3-5) by assuming

$$\begin{aligned} U(x,t) &= \sum_{N=0}^{NT} U_N(x) e^{iN\omega t} \\ \theta(x,y,t) &= \sum_{N=0}^{NT} \theta_N(x,y) e^{iN\omega t} \end{aligned} \tag{12}$$

where NT is the number of harmonics considered. The modified theory replaces Eq. 6 in the general theory by Eq. 12 for the expressions for the displacement $U(x,t)$ and the temperature profiles $\theta(x,y,t)$ in the beam. Substituting Eq. 12 into Eq. 5 gives

$$\theta_N(x,y) = \frac{\Delta_E}{\alpha}\frac{\partial^2 U_N(x)}{\partial x^2}\left[y - \frac{\sin(k_N y)}{k_N \cos(bk_N/2)}\right] \tag{13}$$

for $N > 0$, where $k_N = (iN\omega/\kappa)^{1/2}$, $\Delta_E = E\alpha^2 T_0/\hat{C}_p$ and $\theta_0(x,y) = 0$. I_T can be computed from Eqs. 4 and 13 as

$$I_T = \sum_{N=1}^{NT} \frac{I\Delta_E}{\alpha}\left[1 + f(N\omega)\right]\frac{\partial^2 U_N(x)}{\partial x^2} e^{iN\omega t} \tag{14}$$

which is substituted into Eq. 3 to obtain a frequency dependent Young's modulus $E_{N\omega}$ for each of the harmonics $N > 0$ as

$$E_{N\omega} = E\left[1 + \Delta_E\{1 + f(N\omega)\}\right] \tag{15}$$

and $E_{0\omega} = E$. $f(N\omega)$ in Eqs. 14 and 15 is given by

$$f(N\omega) = \frac{24}{b^3 k_N^3}\left[\frac{bk_N}{2} - \tan\left(\frac{bk_N}{2}\right)\right] \tag{16}$$

The stiffness and the damping constant of each of the harmonics K_N and C_N can be computed from the real and the imaginary part of the corresponding $E_{N\omega}$, respectively, using

$$K_N + iN\omega C_N = pE_{N\omega} \tag{17}$$

where p is a constant that depends on the type of beam (for example $p = 8EI/l^3$ for a cantilever beam of length l [**12**]). While the value of K_N is found to be negligibly affected by the thermoelastic effects and can be assumed to be equal to $K_N = K = pE$ (for example $8EI/l^3$ for a cantilever beam) for all values of N, the value of C_N is found to be

$$C_N = \frac{K\Delta_E}{N\omega}\left(\frac{6}{\xi_N^{\ 2}} - \frac{6}{\xi_N^{\ 3}}\frac{\sinh\xi_N + \sin\xi_N}{\cosh\xi_N + \cos\xi_N}\right) \tag{18}$$

for $N > 0$ and where $\xi_N = b[N\omega/(2\kappa)]^{1/2}$ (Note that $C_0 = 0$). At this point, a mass-spring-damper (MSD) model of MEMS is introduced into the modified theory (to compute the overall damping constant and quality factor) and is given by

$$M\ddot{u} + C\dot{u} + Ku = F_e = \frac{\varepsilon A V^2}{2(g-u)^2} \tag{19}$$

where M is the mass of the beam, C and K are the overall damping and stiffness constants of the beam, respectively, and F_e is the electrostatic force that depends on the undeformed gap g, the area A of the micro-beam facing the ground electrode, the displacement u, the permittivity of vacuum ε and the applied voltage V. A voltage of the form $V = V_{dc} + V_{ac}e^{i\omega t}$ is considered, where the real and the imaginary parts of $e^{i\omega t}$ corresponds to a cosine or sinusoidal ac excitation, respectively. Harmonic balance analysis [**13**] of Eq. 19 is done by substituting the expression

$$u = \sum_{N=0}^{NT} u_N e^{iN\omega t} \tag{20}$$

into Eq. 19 and the closed form expressions for u_N are obtained by equating the coefficients of $e^{iN\omega t}$ for $N = 0,1,2,\ldots,NT$ to zero. While M and K are known in the left hand side of Eq. 19, the thermoelastic damping force can be expressed in terms of C_N as

$$C\dot{u} = \sum_{N=0}^{NT} iN\omega C_N u_N e^{iN\omega t} \tag{21}$$

The closed form expression for u_N for $N > 2$ is given by

$$\begin{aligned} u_N = -[&-M\omega^2 \sum_{I=1}^{R} u_I^2 u_{P-2I}(P-2I)^2 \\ &- 2M\omega^2 \sum_{I=1}^{P-1} u_{P-I}(P-I)^2 \mathrm{T}(I) \\ &+ i\omega \sum_{I=1}^{R} C_{P-2I} u_I^2 u_{P-2I}(P-2I) \\ &+ i2\omega \sum_{I=1}^{P-1} C_{P-I} u_{P-I} \mathrm{T}(I) + K\sum_{I=1}^{R} u_I^2 u_{P-2I} \\ &+ 2K\sum_{I=1}^{P-1} u_I \mathrm{T}(P-I) - 2Kg\Gamma(P) - LKgu_{P/2}^2] \\ &\times [Mu_0^2\omega^2P^2 - iC_P\omega u_0^2 P - 3Ku_0^2 + 2Kgu_0]^{-1} \end{aligned} \tag{22}$$

where $R = P/2-1$ and $L = 1$ when P is even and $R = (P-1)/2$ and $L = 0$ when P is odd. $T(I)$ is the summation of the combination of all possible pairs $u_J u_K$ for $J + K = I$ and $J < K$. u_0 is obtained in a closed form from the solution of the equation

$$(g-u_0)^3 - g(g-u_0)^2 + \frac{\varepsilon A V_{dc}^2}{2K} = 0 \tag{23}$$

and u_1 and u_2 are obtained as

$$\begin{aligned} u_1 &= -\frac{\varepsilon A V_{dc} V_{ac}}{(Mu_0^2\omega^2 - 3Ku_0^2 + 2Kgu_0 - iC_1u_0^2\omega)} \\ u_2 &= \frac{2Mu_0u_1^2\omega^2 - 3Ku_1^2u_0 + Kgu_1^2 - \varepsilon A V_{ac}^2/2}{(4Mu_0^2\omega^2 - 3Ku_0^2 + 2Kgu_0 - i2C_2u_0^2\omega)} \end{aligned} \tag{24}$$

The overall damping coefficient can be computed from the total amount of energy dissipated per period ΔE using C and the values of C_N for N = 0 to NT, and equating them as

$$C = \sum_{N=0}^{NT} C_N \bar{u}_N^2 N^2 \left[\sum_{N=0}^{NT} \bar{u}_N^2 N^2 \right]^{-1} \tag{25}$$

where $\bar{u}_N$ is the absolute value of u_N. The quality factor of the system can be computed using the relation

$$Q_{TED} = \frac{2\pi E_{\max}}{\Delta E} = \frac{K\bar{u}_{\max}^2}{\omega \sum_{N=0}^{NT} C_N \bar{u}_N^2 N^2} \tag{26}$$

where $\bar{u}_{max}$ is the maximum absolute value of u in a period, computed from Eq. 20.

RESULTS

A silicon cantilever beam of dimensions: length = 200 µm, thickness = 5 µm and width = 10 µm, is considered and the effect of electrostatic actuation on the thermoelastic damping coefficient and quality factor of the beam is studied here by comparing the results obtained from the general theory and the modified theory. The material properties used for the beam are given in [7]. The resonant frequency of the beam is found to be 138 KHz and the dynamic pull-in voltage is 44.5 V from the MSD model. Considering the expression for the electrostatic force F_e given in Eq. 19, when the applied voltages are small (far off from pull-in), F_e can be written as

$$F_e \approx \frac{\varepsilon A}{2g^2} \left[V_{dc}^2 + \frac{V_{ac}^2}{2} + 2V_{dc}V_{ac}\sin(\omega t) - \frac{V_{ac}^2 \cos(2\omega t)}{2} \right]$$

for sinusoidal excitation as $u \ll g$. Defining $r = V_{ac}/(V_{dc}+V_{ac})$, when the dc bias is dominant (dc operation) $r \rightarrow 0$ and when the ac voltage is dominant and the dc bias is zero (ac operation), r = 1. Fig. 1 shows the variation in QTED with the excitation frequency for (a) dc operation and (b) ac operation in the MEMS cantilever beam. Zener's model evaluated at $f = \omega/(2\pi)$ (the excitation frequency) matches with the modified theory for dc operation (where the motion is simple harmonic). For ac operation, Zener's model evaluated at $2f$ matches with the modified theory as the oscillations are still simple harmonic but at twice the excitation frequency.

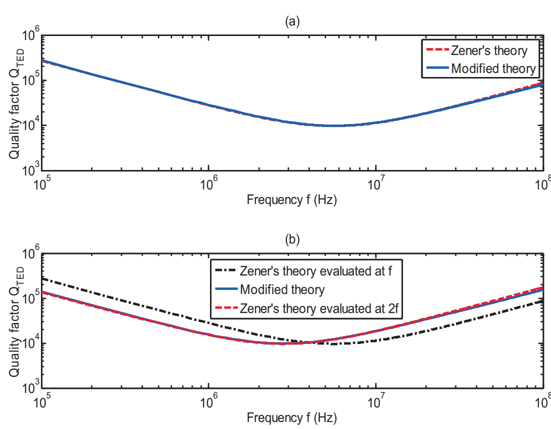

***Figure 1**. Comparison between the general and the modified theories under small applied voltages for (a) dc operation ($r \rightarrow 0$) and (b) ac operation (r=1).*

However, for intermediate values of r between 0 and 1, the electrostatic force F_e contains both the first and the second harmonic components of the excitation frequency. As a result, the oscillations also have both these two components in them giving rise to non simple harmonic oscillations for which case, the general theory of TED is expected to fail. The actual thermoelastic damping constant C and quality factor Q_{TED} in that case can be predicted by the modified theory as shown in Fig. 2.

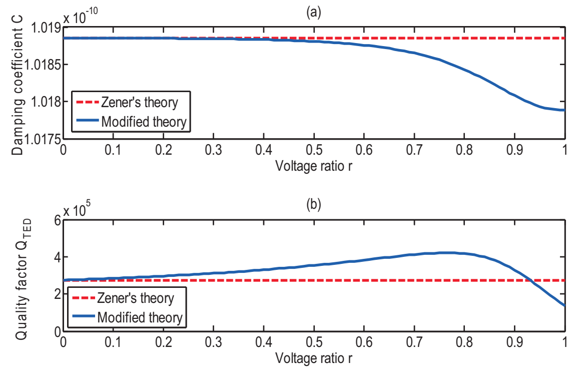

***Figure 2.** Variation in thermoelastic damping coefficient C and Q_{TED} in the MEMS beam at f = 100 KHz with r predicted by the general and the modified theories .r is varied by simultaneously increasing V_{ac} (.001 to 1) and decreasing V_{dc} (1 to 0) linearly.*

When the applied voltages are large (close to pull-in), the nonlinearity in the electrostatic force due to $F_e \propto 1/(g-u)^2$ becomes important and can give rise to complex oscillation [11]. These complex oscillations can significantly alter the values of the damping coefficient and quality factor as shown in Fig. 3 for the MEMS beam. The modified theory shows several spikes in the value of C and Q_{TED} as it varies with f at 42 V dc and 2.8 V ac. This is due to the presence of higher order harmonics (see Fig. 4) in the oscillations introduced by the $F_e \propto 1/(g-u)^2$ nonlinearity at these large voltages. Zener's model does not capture any of these spikes.

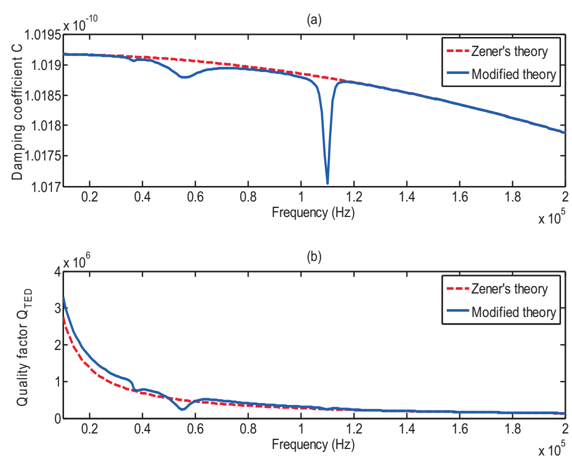

***Figure 3.** Variation in thermoelastic damping coefficient C and Q_{TED} in the MEMS beam with the ac excitation frequency f at a large dc bias of 42 V (near pull-in) and a large ac voltage of 2.8 V (near ac pull-in).*

Fig. 4(a) shows the harmonic balance analysis of the MSD model indicating the presence of several higher order harmonics in the oscillations of the MEMS beam at 42 V dc and 2.8 V ac. The

N-th harmonic is found to spike at the N-th superharmonic frequency other than at the resonant frequency (110 KHz at 42 V dc). As the different harmonics have different damping coefficients C_N (which are evaluated in the modified theory and shown in Fig. 4(b)), the overall damping coefficient/quality factor changes with the variation in the relative strengths of the different harmonics (the first ten harmonics were considered in our analysis).

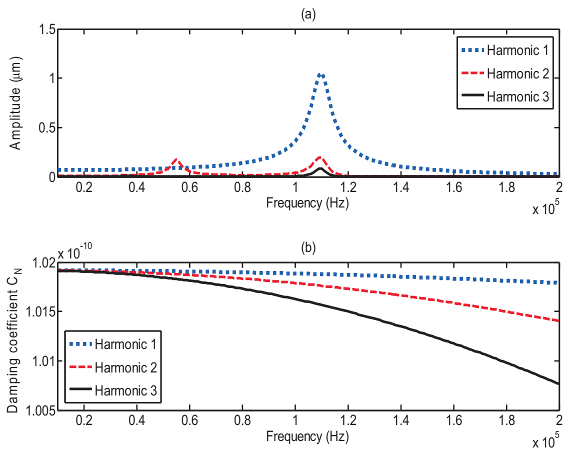

Figure 4. Variation in (a) strength of the harmonics with the excitation frequency from the harmonic balance analysis of the MSD model and (b) CN with the excitation frequency obtained from the modified theory.

As the ac voltage is gradually increased (from 0.001 V ac to 2.5 V ac) under the 42 V dc bias, the difference between the two theories start to become significant as shown in Fig. 5. In fact at 0.001 V ac, the modified theory and the general theory gives the same values of C as the higher order harmonics are not significant at such small ac voltages. As the ac voltage increases, the higher order harmonics become stronger and the modified theory deviates from the general theory of Zener/Lifshitz and Roukes (spikes are formed at the superharmonics of the excitation frequency).

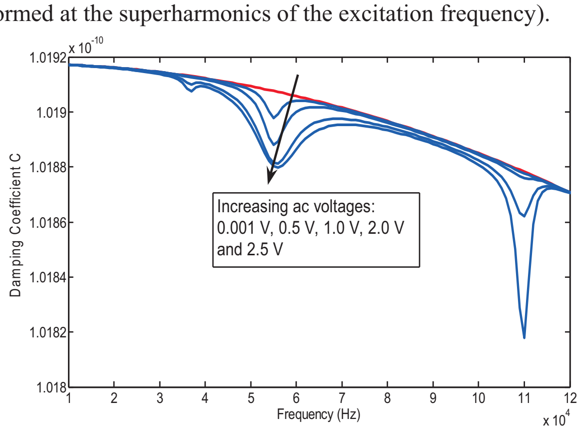

Figure 5. *Deviation of the modified theory from the general theory as the ac voltage increases at 42 V dc in the MEMS cantilever beam. As the ac voltage increases, the higher order harmonics in the oscillations become stronger forming the spikes in variation of C with the excitation frequency f.*

CONCLUSIONS

The general theory of thermoelastic damping is modified in this paper for application to electrostatic MEMS. The nonlinear electrostatic force present under electrostatic actuation can give rise to complex oscillations in the system which cannot be accounted for by the general theory (based on simple harmonic oscillations). The modified theory takes into account these higher order harmonics present in the oscillations due to nonlinear electrostatic force to compute the overall damping coefficient and quality factor accurately.

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