A HEURISTIC APPROACH TO THE ELECTROMECHANICAL MODELING OF MEMS BEAMS

Michael R. Boyd, Selden B. Crary, and Martin D. Giles

Center for Integrated Sensors and Circuits
EECS Department, The University of Michigan
1301 Beal Avenue, Ann Arbor, Michigan 48109-2122
Contacts: crary@umich.edu or madagil@umich.edu

ABSTRACT

Approaches for calculating the capacitance of parallel plate capacitors of a variety of shapes and plate spacing are presented. Accurate solutions based on the method of subareas is compared to a new, geometry-based approximation which can be rapidly evaluated and shows excellent agreement over a wide range of capacitor configurations. The new approach reduces the computational cost of coupled electromechanical calculations of MEMS structures by essentially eliminating the cost of capacitance calculation in solving the coupled system until the final iteration.

INTRODUCTION

Many MEMS structures utilize beams driven electrostatically. The approach advanced for the modeling of such structures, which can have substantial fringing-field effects, is coupled finite- and/or boundary-element electrostatic and mechanical calculations[1]. Such calculations incur a computational burden that should be minimized as far as possible. It is of interest and importance to find means of expressing the capacitance of MEMS beam structures in simple but accurate mathematical expressions that can be rapidly evaluated. Such expressions could be used directly for capacitance estimation. They could also be used in the initial iterations of the coupled-field solution of electromechanical systems to reduce computation time, switching to a more conventional approach for the final iteration where necessary.

Here we present approaches to the calculation of capacitance for parallel-plate capacitors of various plate shapes and plate spacings in vacuum. 3D capacitance calculations can be made directly, or approximate expressions derived by extension of 2D calculations for infinite parallel-plates. All capacitance calculations scale according to a single length parameter, so the results can be applied to structures at any length scale. Normalized results can be obtained by dividing by the capacitance of an equivalent parallel plate capacitor neglecting fringing fields.

CALCULATING CAPACITANCE IN 2D

The mutual capacitance per unit length of a pair of infinitely-long parallel plates, or, by a familiar application of the method of images, of a single infinitely-long plate above an infinite ground plane, can be obtained exactly using the Schwartz-Christoffel transformation [2]. The capacitance in either case is given by

$$C_{Palmer} / \varepsilon = K' / K$$

where $K(k)$ is the complete elliptic integral of the first kind, and $K'(k) = K(1-k^2)$ is the complementary complete elliptic integral. The parameter k is an implicit function of the ratio of plate width to plate gap, $R = w / g$,

$$R = \frac{2}{\pi}\left(K'E'(\beta,k) - E'F'(\beta,k)\right) \tag{1}$$

$$\sin^2\beta = \frac{K' - E'}{(1-k^2)K'}$$

where $E'(\beta,k)$ is the complementary elliptical integral of the second kind and $F'(\beta,k)$ is the complementary elliptic integral of the first kind. Since Equation 1 is not invertable, calculation for a particular value of R requires an iterative or table-lookup procedure to determine the corresponding value for k. For large values of R, an approximate capacitance formula has been derived by Love[3] effectively based on a series expansion of the elliptic integrals. This results in the expression:

$$C_{Love} / \varepsilon = R + \frac{1}{\pi}(1 + \ln 2\pi R) \tag{2}$$

This gives capacitance values within 3% of the exact value for $R > 3$.

The zeroeth-order approximation for the capacitance per unit length is simply $\varepsilon w/g$. Deviations from this value are called fringing-field corrections. Figure 1 shows the effect of fringing fields on the capacitance of the two beams, based on eleven evenly spaced values of the gap-to-width ratio, using the exact method. The data are labeled Exact and MSA Beams.

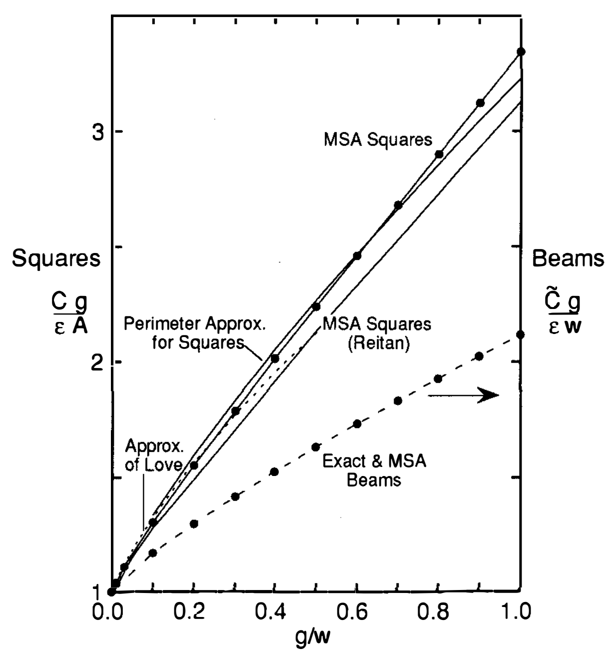

Figure 1. **(Exact & MSA Beams)** The mutual capacitance per unit length of a pair of infinitely long parallel plates, normalized by the mutual capacitance neglecting fringe-field effects $\varepsilon w/g$, is shown plotted as a function of the gap-to-width ratio g/w. Straight-line segments connect points calculated using the exact method. Filled circles represent values calculated using the method of subareas. **(MSA Squares)** The mutual capacitance of a pair of square plates, normalized by the mutual capacitance neglecting fringe-field effects $\varepsilon A/g$ is shown in the same manner. **(Perimeter approx.)** The approximate formula for the capacitance of planar structures using the area and perimeter terms of the geometric expansion of Eq. 7 provides a good approximation to the actual capacitances. **(Approx. of Love)** Love's approximation [3] provides a series-based alternative to the perimeter approximation. **(MSA Squares - Reitan)** The data of Reitan using a 6x6 mesh [4] are shown for comparison.

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An alternative to transformation-based methods is the method of subareas (MSA) described by Reitan[4]. This scheme subdivides the plates into a number of small subareas such that the charge density is essentially constant over each subarea. In 2D, the potential at subarea j due to charge on subarea i can be written

$$V_{ij} = \frac{-q_i}{2\pi\varepsilon} \int_{-h/2}^{h/2} \ln\left(\sqrt{(x+\Delta x)^2 + (\Delta y)^2}\right) dx \tag{3}$$

where q_i is the charge per unit length on subarea i, h is the subarea width, and $\Delta x, \Delta y$ are the distances between the centers of the subareas along the x and y axes. This integral holds even when $i = j$. The total potential at subarea i is then given by

$$V_i = \sum_{j=1}^{n} V_{ij} \tag{4}$$

For a charged conductor at equilibrium, the potential is constant across the entire surface, $V_i = V_0$. In that case, Equation 4 represents a set of n simultaneous linear equations which can be solved for the individual subarea charges. By symmetry, the top and bottom plates must have equal magnitude charge distributions but opposite polarity, and charge distribution for each plate must be symmetric about the center line. Consequently, charges for only a quarter of the total subareas need be explicitly calculated. The overall capacitance is given by $C = Q / V_0$, where Q is the total charge on one plate.

For a gap-to-width ratio of 0.1, uniform discretizations with numbers of subdivisons m across each plate of 2, 10, 20, 30, 100, 200, 300, 1000 were used. A convergence analysis indicated that the discretization error converged to the following power law with exponent β=-1.00:

$$C - C_0 = \alpha h^{\beta}, \tag{5}$$

where α is a constant determined by curve fitting. The convergence is demonstrated graphically in Figure 2.

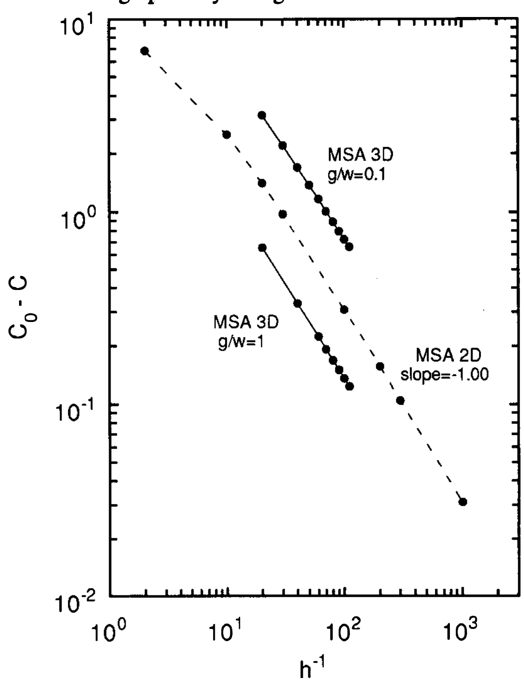


Figure 2. The results of the 2D MSA computations follow a power-law behavior with exponent -1.00, in the limit of small subdivision size h. The results of the 3D computations have not yet reached the region of asymptotic slope by h^{-1}=110, although a detailed analysis shows that the slopes themselves are converging via a power law to asymptotic values near -1.00.

Figure 1 shows as filled circles the results of eleven MSA calculations made for the same eleven values of g/w as were used for the exact calculations. The number of subdivisions across each beam was 1000. The convergence analysis indicated that these values could be used without power-law extrapolation and were accurate to 0.02%. Extrapolation via the power law could be used to reduce the error to below 0.01%.

The MSA can also be used to calculate the capacitance to infinity of a single plate in a similar manner.

CALCULATING CAPACITANCE IN 3D

The only three-dimensional capacitance formula known is that for the capacitance to infinity of an elliptical disk. There are no formulae available for finite, parallel-plate capacitors. Approximate solutions can be obtained by applying the method of subareas in 3D. For square subareas, Equation 3 becomes

$$V_{ij} = \frac{-q_i}{4\pi\varepsilon} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \left((x+\Delta x)^2 + (y+\Delta y)^2 + (\Delta z)^2\right)^{-1/2} dx\, dy$$

where q_i is the charge on subarea i. A set of linear equations for the subarea charges is formed as before and the system solved to yield the subarea charges. Square plates have eight-fold symmetry, so only 1/16 of the total number of subareas need be explicitly calculated in that case. The rectangular and L-plate examples presented later have reduced symmetry and so require explicit calculation of a larger fraction of subareas.

Because 3D MSA computations can be expected to require many more subareas than their 2D counterparts, we made an initial set of 3D MSA runs to determine the convergence behavior for square plates with gap-to-width ratio of 0.1 at a set of increasingly fine uniform discretizations. Each square was divided into $m \times m$ subareas, and computations were made with m ranging from 20 to 110 in steps of 10. A power-law fit to the trio of results using m=90, 100, and 110 gave a power law of β=-0.9401; a fit using m=80, 90, and 100 gave β=-0.9339; and a fit using the trio m=70, 80, and 90 gave β=-0.9261. In a similar fashion, fits to the exponent for gap-to-width ratio of 1.0 gave β=-0.9776, β=-0.9759, and β=-0.9739, respectively. To test if the exponents for these two gap-to-width ratios were converging, we then performed power-law fits of the convergence of the β's, under the following assumed power law:

$$\beta - \beta_0 = \gamma h^{\delta}, \tag{6}$$

where β_0 is the converged value of β being determined. The fits gave β =-0.998 and -1.007 for gap-to-width ratios of 0.1 and 1.0, respectively.

We then tentatively assumed that the asymptotic value for β was unity in this 3D case, as it was in the 2D case, and we performed a variety of checks of the consistency and adequacy of this assumption. An integration of Equations 5 and 6 gave the following formula for the converged value of the capacitance C_0, based on the capacitances computed for the five finest meshes:

$$C_0 = \frac{C_3 - C_1 e^S}{1 - e^S} \quad ,$$

where C_3 is the capacitance computed for the middle (h_3) of the three finest meshes of the five, C_1 is the capacitance computed for the middle (h_1) of the three coarsest meshes of the five, S is given by

$$S = \beta_0 \ln\frac{h_1}{h_3} + \frac{\gamma}{\delta}(h_1^{\delta} - h_3^{\delta}) \quad ,$$

and all the other constants are determined in the course of fitting to Equation 6. This procedure gave C_0=1.3059 for a gap-to-width ratio of 0.1 and C_∞=0.33429 for a gap-to-width ratio of 1.0. Figure 2 shows the discretization error as a function of the length of a subdivison of the discretization, on the assumption that the converged values C_0 are 1.3059 and 0.33429 for gap-to-width ratios of 0.1 and 1.0, respectively.

Under the assumption that β=1, a simpler extrapolation from the finest two discretizations, that is, from the computations at m=100 and 110, gave C_∞=115.59, and an extrapolation from the computations made at m=40 and 60 gave C_0=115.61. Our conclusion is that extrapolations can be made from the moderately coarse discretizations of n=40 and 60 using an assumed value of β=1, and that the error introduced in this way is less than 0.1%. The extrapolation method assuming β=1 was then applied to computations for squares with gap-to-width ratios ranging from 0.1 to 1.0 for discretizations of m=40 and m=60. The results are shown in Table 1.

Figure 1 includes a plot of the mutual capacitance of two parallel square plates, or, equivalently, of a single square plate above an infinite ground plane, as a function of the gap-to-width ratio, based on the computations and extrapolations discussed above. The normaliztion factor is the zeroeth-order capacitance $\varepsilon A/g$, where A is the area of one plate, $A=w^2$. The effect of fringing in the 3D case is roughly double that observed in 2D.

To verify the accuracy of the 3D MSA code, two calculations with known solutions were made. First, we considered the capacitance to infinity of a circular disc of radius r, which is known to be $C = 8\varepsilon r$. Approximating the circle using a square grid with up to 110 subareas across the diameter, convergence analysis yields a capacitance of 35.337 pF for a 1 meter diameter, compared to an exact solution of 35.417 pF. Second, we considered the capacitance between two long parallel plates. As the length of the plates increases, the capacitance per unit length approaches the exact 2D result for infinite parallel plates, as shown in Figure 3.

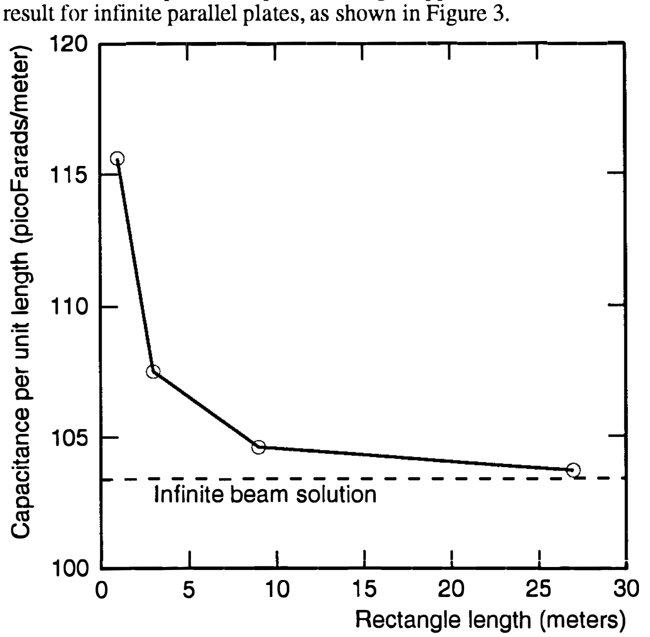


Figure 3. The capacitance per unit length of rectangular beams converges to the capacitance of an infinite beam, in the limit of the long side of the rectangle going to infinity, while holding the length of the short side constant.

Our 3D MSA code is capable of calculating capacitance for a number of different conductor shapes including rectangles, rectangles with a rectangular hole in the center, and rectangles with one quadrant removed. A plot of charge density across a 1 meter square plate of a parallel-plate capacitor with 0.1 meter gap is shown in Figure 4. The square has a 0.5 meter square hole in the center, which appears as a region of zero charge density in the plot. This clearly shows the components of the charge distribution of the capacitor: a constant value over the area of the plate, an additional edge charge around the perimeter of the plate, a further increase in charge at exterior corners, and a decrease in charge at interior corners. These components can also be seen in the cross-sections of Figure 5.

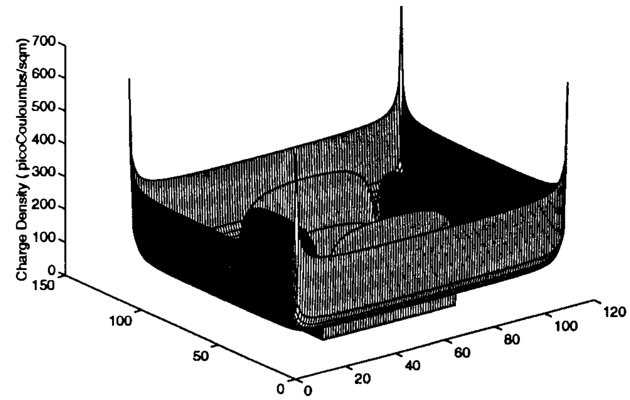


Figure 4. 3D plot of the charge density on one of two parallel square plates with holes in their centers. The plot is based on the average charge over each subarea and thus shows effects of discretization. Nonetheless, the very strong singularity in charge density at the exterior corners is observed, as are the decrease in charge density at interior corners and the singularity at the edges.

GEOMETRIC EXPANSION

The association of capacitor charges with geometic features of the plates suggests an approach to rapidly estimating capacitance based on geometry:

$$C = A\,C_{area} + P\,C_{perim} + E\,C_{ext} + I\,C_{\text{int}} + \ldots \tag{7}$$

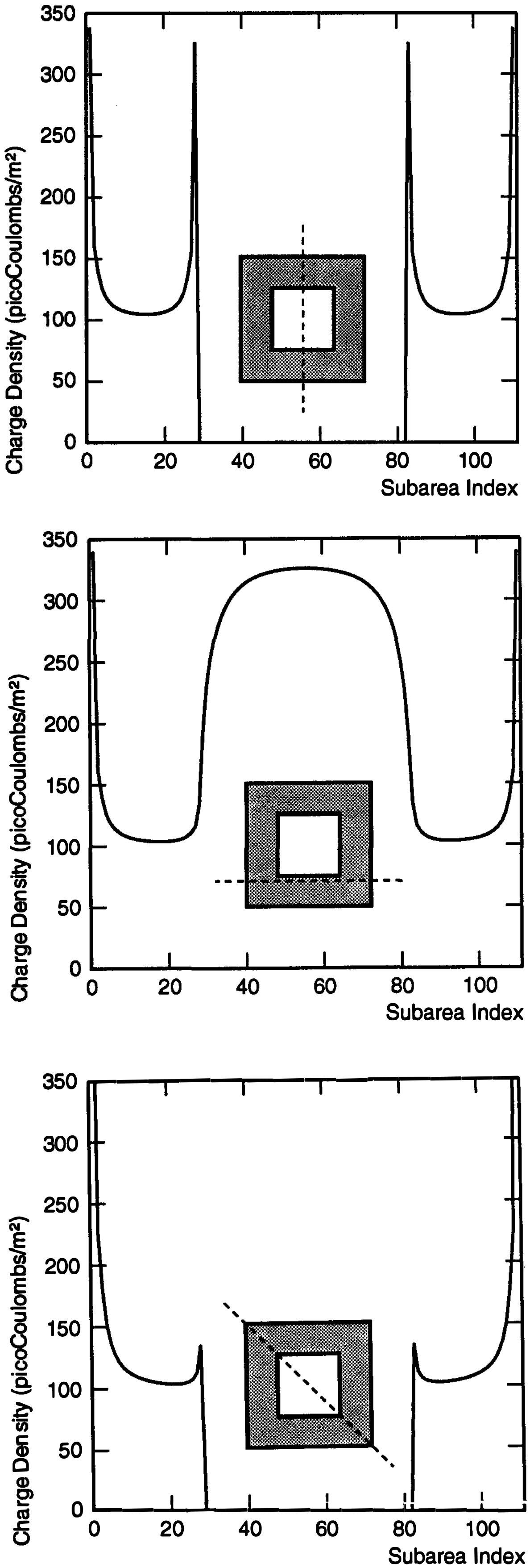
where A is the plate area, P is the perimeter length, E is the number of exterior corners, and I is the number of interior corners. The first term can be identified with the ideal capacitance expression, $C_{area} = \varepsilon / g$. The second term can be obtained from the known capacitance of infinite parallel plates per unit length:

$$C_{Palmer} = W\,C_{area} + 2\,C_{perim} \tag{8}$$

For hand calculations, the approximate formula of Love (Eq. 2) can also be used. Neglecting further terms in the geometric expansion, we can compare capacitances calculated using this approach with the results of MSA for parallel plate capacitors with a variety of shapes. Summaries of the results are presented in Tables 1 and 2. The simple geometric approach of Eq. 8 shows excellent agreement over a wide range of plate shapes and plate gaps.

Gap	2D Exact	3D Geometric approximation	3D MSA 7200 subareas	3D MSA extrapolated	Error
0.1	103.414	118.29	114.461	115.53	2.4%
0.2	57.470	70.67	68.089	68.71	2.9%
0.3	41.787	54.06	52.230	52.69	2.6%
0.4	33.785	45.43	44.163	44.54	2.0%
0.5	28.895	40.08	39.267	39.59	1.2%
0.6	25.581	36.40	35.979	36.27	0.4%
0.7	23.176	33.70	33.621	33.89	-0.6%
0.8	21.346	31.62	31.850	32.10	-1.5%
0.9	19.903	29.97	30.473	30.70	-2.4%
1.0	18.734	28.61	29.375	29.59	-3.3%

Table 1. Square 1x1 parallel plate capacitor calculations for a variety of plate gaps. For plate dimensions in meters, the capacitance values are in picoFarads. MSA linear extrapolation was based on 3200 and 7200 subarea calculations. The 2D Exact calculation uses Eq. 1 for infinite parallel plates of width 1 and corresponding length.



Figures 5. This figure shows the charge density across one of the two square plates with a hole in it for three different cuts, as indicated.

Shape	3D Geometric approximation	3D MSA	3D MSA extrapolated	Error
1x1 square	118.3	114.24	115.5	2.4%
3x1 rectangle	325.6	319.25	322.5	1.0%
9x1 rectangle	946.8	933.98	941.7	0.5%
2x2 L-shape	325.6	317.13	321.4	1.3%
3x3, 1x1 hole	828.3	812.38	821.3	0.9%

Table 2. Parallel plate capacitor calculations for a variety of plate shapes with plate gap 0.1. For plate dimensions in meters, the capacitance values are in picoFarads. MSA calculations used 5000 subareas and linear extrapolation between the 3200 and 5000 subarea calculations, except for the square with a hole which used 5184 and 7744 subareas.

We have also made comparisons in Figure 1 with the MSA results of Reitan [4] that used a 6x6 discretization and with the approximate equation of Love that is valid for $g/w<<1$. Finally, we made several runs using FASTCAP [5]. For beams with length 100 and width 1, the FASTCAP results were consistently low by a large margin. FASTCAP runs for a variety of squares also gave results that were also consistently low. For example, our result from a FASTCAP run for a 1x1 square with a gap of 0.1 agreed with the FASTCAP User's Guide [5], but was low by 10% compared to our 3D MSA runs.

CONCLUSION

Geometric-based electrostatic calculations provide a simple and rapid means of calculating capacitances for MEMS, at least for the simple flat geometries discussed in this paper. Future research should be able to extend these concepts to include more realistic MEMS geometries that include thicknesses and rounded corners.

For self-consistent coupled electrostatic and mechanical modeling, the approximate expressions for capacitance presented here can be used in at least two ways. For initial exploration of system behavior, they provide a much quicker solution than a detailed capacitance calculation. For more accurate calculations they can be used in the initial iterations of the coupled system, switching to a more detailed calculation when the system is close to convergence.

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